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A Fokker-Planck model for wealth inequality dynamics

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Abstract – Studying the mechanisms that govern the dynamics of the wealth distribution is essential for understanding the recent trend of growing wealth inequality. A particularly important explanation is Piketty's argument, giving credit to the seminal events of the first half of the 20th century for the relatively egalitarian second half of this century. Piketty suggested that these dramatic events were merely a perturbation imposed on the economy affecting the wealth structure, while in general, wealth inequality tends to increase regularly. We present a simple stochastic model for wealth and income based on coupled geometric Brownian motions and derive a Fokker-Planck equation from which the joint wealth-income distribution and its moments can be extracted. We then analyze the dynamics of these moments and hence of the inequality. Our analysis largely supports Piketty's argument regarding the irregularity of the 20th century, that wealth inequality inevitably tends to increase. We find, however, that even if wealth inequality will eventually go up, under plausible conditions, it can go down for periods of up to several decades.

Introduction. – The growing wealth inequality in most western countries during the past several decades led to an increased interest in the nature of the wealth inequality dynamics. An important argument in that context, recently made by Piketty [1] and Scheidel [2], was that an increase in wealth inequality is inevitable and that the decrease in wealth inequality during half of the 20th century is, to a large extent, a singular phenomenon.

Income inequality had also increased dramatically in the past few decades in many countries. Wealth and income inequality are related and are positively correlated, but they can differ considerably. The difference between the distributions of wealth and income and the imperfect correlation between them, was found as important for the dynamics of the wealth inequality [3,4]. The correlation between income and wealth in the US, for example, is about 0.5 [5,6]. This indicates that there are wealthy individuals who earn relatively low incomes, and on the other hand high income receivers with low wealth. This partially explains the difference between income and wealth inequalities [6].

Different modeling approaches were suggested in the past few decades in order to better understand the sources of wealth inequality [3,7–12]. These approaches, in addition to claims raised in response to Piketty's argument, are debated on whether wealth inequality generally tends to increase indefinitely, making wealth concentrated more and more in the hands of few. The time scales of processes governing the wealth distribution are also debatable and there is no consensus regarding which elements in the economy are affecting these processes.

The purpose of this paper is to analyze the dynamics of wealth and income and in particular, to determine under which circumstances wealth inequality increases and decreases. We present a simple stochastic model for wealth and income based on coupled geometric Brownian motions (GBMs) and derive a Fokker-Planck equation from which the joint wealth-income distribution and its moments can be extracted. We then analyze the dynamics of these moments and hence of the inequality, quantified by the coefficient of variation (CV). Despite its simplicity, the model can be used for describing the important aspects of wealth inequality dynamics. Simpler models would not be able to capture such dynamics.

The model substantially differs from previous models of wealth dynamics, in particular refs. [3,10]. In this paper we present a much generalized model with two substantial improvements:

– The model includes explicit stochastic terms and the treatment of the joint income-wealth dynamics. In previous models, such as those presented in refs. [3,10], income was taken as an external contribution to wealth and the dynamics were deterministic.

– The way our model is now formulated allowed us to provide quantitative closed-form expressions for the wealth distribution moments. These enable, in particular, calculating a criterion for the initial decrease of wealth inequality and calculating characteristic relaxation times – the time it takes wealth inequality to return to its initial level following its initial decrease. These had only been done previously using numerical simulations.

Our analysis largely supports Piketty's arguments regarding the irregularity of the 20th century and that wealth inequality inevitably tends to increase. We find, however, that under certain conditions, namely, a low correlation between wealth and income, and a low wealth-income ratio (the ratio between the total wealth accumulated by individuals in an economy and the aggregated income of all individuals for a given year), a temporary decrease in wealth inequality is possible. This decrease can last for several decades under plausible parameter values. The wealth-income ratio had high values of 6–7 in Europe in the 18th–19th centuries, came down to 2–3 in 1970 and since then has gone up to 4–6 [13]. It is very central in Piketty's theory, which is largely based on the relationship between this ratio and wealth inequality, which our analysis reinforces.

A stochastic model for the dynamics of wealth and income. – The starting point of the model is the following set of equations for the dynamics of the individual wealth (x_1) and saved income (x_2 —which we will refer to as the income, for simplicity):

$$\begin{aligned} \dot{x_1} &= (\lambda_1 + \eta_1)\, x_1 + a_1 x_2, \\ \dot{x_2} &= (\lambda_2 + \eta_2)\, x_2 + a_2 x_1, \end{aligned} \tag{1}$$

where λ_1 is an average rate of return, λ_2 is the income growth rate, and η_1 and η_2 are the return and growth stochastic noise, respectively, both delta correlated in time (with $\sigma_1^2\delta(t)$ and $\sigma_2^2\delta(t)$). a_1 and a_2 are the coupling coefficients between wealth and income.

Practically, a_1 will be regarded as 1 and a_2 as 0, since we consider the income x_2 to be after taxes and after spending (the income includes earnings, dividends and other types of capital income [3]). In addition, we assume that the income follows a geometric Brownian motion, in which the stochastic term represents changes in consuming habits, changes in skill and education, promotion and dismissal. Equation (1) also follows a GBM, but with an external force (the saved income), resembling a Langevin equation. Treating wealth and income as GBMs is a standard approach, based on the multiplicative nature of economic processes, such as inflation, interest, economic growth and investments, which govern the dynamics of both income and wealth [7,11,14].

However, at the same time, the model is a simplified description of the dynamics of wealth and income, making the following assumptions:

– The noise terms for both wealth and income (η_1 and η_2) are considered as independent between individuals.

– The wealth and income growth rates (λ_1 and λ_2) are identical for each individual. They are also considered as independent on the individual wealth and income.

– No explicit exchange of income or wealth between individuals and no "economic structure" (*e.g.*, taxation), which creates interactions between individuals, are considered.

– As put by Meade [15], our agents "do not marry or have children or die or even grow old" ([15], p. 41). Therefore, inheritance is not considered as well.

To be clear, we do not wish to separate different processes that affect the wealth distribution, as would be useful if we wanted to understand the effect of a specific process, say inheritance tax, on economic inequality. We aim for the opposite —a model that captures in as few parameters as possible the dynamics of the wealth distribution. Treating the above aspects explicitly would likely affect the model results. Nevertheless, despite these simplifications, implicitly incorporating these effects in the noise term, makes the model capable of describing the important aspects of the wealth distribution changes in time.

Model analysis. Equation (1) can be rewritten as

$$\dot{x_i} = \eta_i x_i + \sum_{j=1,2} a_{i,j} x_j, \tag{2}$$

where

$$A = \begin{pmatrix} \lambda_1 & a_1 \\ a_2 & \lambda_2 \end{pmatrix}. \tag{3}$$

Based on these equations and assuming the Stratonovich convention we can write the corresponding Fokker-Planck equation for the joint wealth and income distributions:

$$\frac{\partial P\,(x_1, x_2, t)}{\partial t} = \sum_{i=1,2}\left[\frac{\sigma_i^2}{2}\frac{\partial}{\partial x_i} x_i \frac{\partial}{\partial x_i} x_i P - \sum_{j=1,2} a_{i,j}\frac{\partial}{\partial x_i} x_j P\right]. \tag{4}$$

Let us use eq. (4) to obtain the moments of the distribution P. By multiplying both sides of eq. (4) by x_1 and integrating with respect to x_1 and x_2 we get

$$\dot{\langle x_1\rangle} = \left(\frac{\sigma_1^2}{2} + \lambda_1\right)\langle x_1\rangle + a_1\langle x_2\rangle. \tag{5}$$

Similarly,

$$\dot{\langle x_2\rangle} = \left(\frac{\sigma_2^2}{2} + \lambda_2\right)\langle x_2\rangle + a_2\langle x_1\rangle, \tag{6}$$

where $\langle\cdot\rangle$ denotes an ensemble average —meaning an average over the population. Since the stochastic processes in eqs. (5), (6) are non-ergodic, we note that the time average (*e.g.*, $\lim_{t\to\infty} 1/T\int_0^T x_1(t)\mathrm{d}t$) will be different from the ensemble average. However, for practical purposes, if large populations are considered (*e.g.*, the population of nation states), these differences are negligible. For a thorough discussion of these effects the reader is referred to refs. [11,16,17].

In order to obtain an equation for the second moments, we multiply both sides of eq. (4) by x_1^2 and integrate with respect to x_1 and x_2. We then get

$$\langle \dot{x_1^2}\rangle = 2\left(\sigma_1^2+\lambda_1\right)\langle x_1^2\rangle + 2a_1\langle x_1x_2\rangle \tag{7}$$

and, similarly,

$$\langle \dot{x_2^2}\rangle = 2\left(\sigma_2^2+\lambda_2\right)\langle x_2^2\rangle + 2a_2\langle x_1x_2\rangle. \tag{8}$$

By applying the same technique to eq. (4), we obtain an equation for $\langle x_1x_2\rangle$:

$$\langle \dot{x_1x_2}\rangle = \left(\frac{\sigma_1^2+\sigma_2^2}{2}+\lambda_1+\lambda_2\right)\langle x_1x_2\rangle + a_2\langle x_1^2\rangle + a_1\langle x_2^2\rangle. \tag{9}$$

We can assemble these equations into a system of first-order linear differential equations:

$$\begin{pmatrix} \langle \dot{x_1}\rangle \\ \langle \dot{x_2}\rangle \\ \langle \dot{x_1^2}\rangle \\ \langle \dot{x_2^2}\rangle \\ \langle \dot{x_1x_2}\rangle \end{pmatrix} = \begin{pmatrix} \frac{\sigma_1^2}{2}+\lambda_1 & a_1 & 0 & 0 & 0 \\ a_2 & \frac{\sigma_2^2}{2}+\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 2\left(\sigma_1^2+\lambda_1\right) & 0 & 2a_1 \\ 0 & 0 & 0 & 2\left(\sigma_2^2+\lambda_2\right) & 2a_2 \\ 0 & 0 & a_2 & a_1 & \frac{\sigma_1^2+\sigma_2^2}{2}+\lambda_1+\lambda_2 \end{pmatrix} \times \begin{pmatrix} \langle x_1\rangle \\ \langle x_2\rangle \\ \langle x_1^2\rangle \\ \langle x_2^2\rangle \\ \langle x_1x_2\rangle \end{pmatrix}. \tag{10}$$

Based on the obtained system, we can now continue to analyze the dynamics of the different moments of the joint distribution of x_1 and x_2, under certain constraints on the model parameters.

The negligible income special case. Initially, we will consider a simplified case in which the income is negligible, similarly to [11]. In this case

$$\langle \dot{x_1}\rangle = \left(\frac{\sigma_1^2}{2}+\lambda_1\right)\langle x_1\rangle \tag{11}$$

and

$$\langle \dot{x_1^2}\rangle = 2\left(\sigma_1^2+\lambda_1\right)\langle x_1^2\rangle. \tag{12}$$

It is now possible to obtain an expression for the dynamics of the wealth distribution coefficient of variation (CV), which we use for quantifying inequality. This measure is one of the commonly used quantitative measures of inequality [6,18,19]. Since we focus on the dynamics of the wealth inequality we consider:

$$CV(t) = \sqrt{\frac{\langle x_1^2(t)\rangle - \langle x_1(t)\rangle^2}{\langle x_1(t)\rangle^2}} = \sqrt{\frac{\langle x_1^2(0)\rangle}{\langle x_1(0)\rangle^2}e^{\sigma_1^2 t}-1}. \tag{13}$$

Equation (13) demonstrates that in this very simplified case, in which income plays no role in the dynamics of wealth, the coefficient of variation only increases in time, and is independent of the rate in which wealth grows. This shows that such a simplified case is inadequate for describing the dynamics of the wealth distribution. Furthermore, it demonstrates that income should be considered, which supports its well-known role in the dynamics of wealth inequality [3,10].

In addition, this analysis demonstrates that in a system with such a random multiplicative noise, inequality will increase "spontaneously". Without noise, inequality would not increase unless other mechanisms are introduced [11,20].

The general case. Let us now consider a more realistic case, in which income cannot be generally neglected in the dynamics of wealth. We assume that $a_1 = 1$ and $a_2 = 0$. Our system of equations becomes

$$\begin{pmatrix} \langle \dot{x_1}\rangle \\ \langle \dot{x_2}\rangle \\ \langle \dot{x_1^2}\rangle \\ \langle \dot{x_2^2}\rangle \\ \langle \dot{x_1x_2}\rangle \end{pmatrix} = \begin{pmatrix} \frac{\sigma_1^2}{2}+\lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \frac{\sigma_2^2}{2}+\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 2\left(\sigma_1^2+\lambda_1\right) & 0 & 2 \\ 0 & 0 & 0 & 2\left(\sigma_2^2+\lambda_2\right) & 0 \\ 0 & 0 & 0 & 1 & \frac{\sigma_1^2+\sigma_2^2}{2}+\lambda_1+\lambda_2 \end{pmatrix} \times \begin{pmatrix} \langle x_1\rangle \\ \langle x_2\rangle \\ \langle x_1^2\rangle \\ \langle x_2^2\rangle \\ \langle x_1x_2\rangle \end{pmatrix}. \tag{14}$$

We first solve the equation for $\langle x_2^2\rangle$:

$$\langle \dot{x_2^2}\rangle = 2\left(\sigma_2^2+\lambda_2\right)\langle x_2^2\rangle, \tag{15}$$

getting

$$\langle x_2^2(t)\rangle = \langle x_2^2(0)\rangle e^{2\left(\sigma_2^2+\lambda_2\right)t}. \tag{16}$$

Now it is possible to solve

$$\langle \dot{x_1x_2}\rangle = \langle x_2^2(t)\rangle + \left(\frac{\sigma_1^2+\sigma_2^2}{2}+\lambda_1+\lambda_2\right)\langle x_1x_2\rangle, \tag{17}$$

for which we obtain

$$\langle x_1 x_2 (t)\rangle = \frac{2\langle x_2^2 (0)\rangle e^{\left(\lambda_2+\frac{\sigma_2^2}{2}\right)t}}{2\lambda_1 - 2\lambda_2 + \sigma_1^2 - 3\sigma_2^2}\left(e^{\left(\lambda_1+\frac{\sigma_1^2}{2}\right)t} - e^{\left(\lambda_2+\frac{3\sigma_2^2}{2}\right)t}\right) + e^{\left(\lambda_1+\lambda_2+\frac{\sigma_1^2+\sigma_2^2}{2}\right)t}\langle x_1 x_2 (0)\rangle. \tag{18}$$

Following these solutions, it is possible to solve also

$$\dot{\langle x_1^2\rangle} = 2\left(\sigma_1^2 + \lambda_1\right)\langle x_1^2\rangle + 2\langle x_1 x_2 (t)\rangle, \tag{19}$$

and obtain

$$\begin{aligned}\langle x_1^2 (t)\rangle &= e^{2\left(\lambda_1+\sigma_1^2\right)t}\left(\langle x_1^2 (0)\rangle + \frac{4\langle x_1 x_2 (0)\rangle}{2\lambda_1 - 2\lambda_2 + 3\sigma_1^2 - \sigma_2^2}\right) \\ &+ \frac{2\langle x_2^2 (0)\rangle e^{2\left(\lambda_1+\sigma_1^2\right)t}}{\left(\lambda_1 - \lambda_2 + \sigma_1^2 - \sigma_2^2\right)\left(2\lambda_1 - 2\lambda_2 + 3\sigma_1^2 - \sigma_2^2\right)} \\ &- \frac{4\langle x_1 x_2 (0)\rangle e^{\left(\lambda_1+\lambda_2+\frac{\sigma_1^2+\sigma_2^2}{2}\right)t}}{2\lambda_1 - 2\lambda_2 + \sigma_1^2 - 3\sigma_2^2} \\ &- \frac{8\langle x_2^2 (0)\rangle e^{\left(\lambda_1+\lambda_2+\frac{\sigma_1^2+\sigma_2^2}{2}\right)t}}{\left(2\lambda_1 - 2\lambda_2 + \sigma_1^2 - 3\sigma_2^2\right)\left(2\lambda_1 - 2\lambda_2 + 3\sigma_1^2 - \sigma_2^2\right)} \\ &+ \frac{2\langle x_2^2 (0)\rangle e^{2\left(\lambda_2+\sigma_2^2\right)t}}{\left(2\lambda_1 - 2\lambda_2 + \sigma_1^2 - 3\sigma_2^2\right)\left(\lambda_1 - \lambda_2 + \sigma_1^2 - \sigma_2^2\right)}.\end{aligned} \tag{20}$$

We also solve the equations for the expected values of x_1 and x_2 (from eqs. (5) and (6)):

$$\langle x_1 (t)\rangle = \langle x_1 (0)\rangle e^{\left(\lambda_1+\frac{\sigma_1^2}{2}\right)t} + \frac{2\langle x_2 (0)\rangle}{2\lambda_1 - 2\lambda_2 + \sigma_1^2 - \sigma_2^2}\left(e^{\left(\lambda_1+\frac{\sigma_1^2}{2}\right)t} - e^{\left(\lambda_2+\frac{\sigma_2^2}{2}\right)t}\right) \tag{21}$$

and

$$\langle x_2 (t)\rangle = \langle x_2 (0)\rangle e^{\left(\lambda_2+\frac{\sigma_2^2}{2}\right)t}. \tag{22}$$

Once again, we consider the coefficient of variation —$CV(t)$ (see eq. (13))— as a measure of wealth inequality. In order to validate the model, we compare the theoretical CV to the measured CV, as reported in ref. [6] for the US economy. The data is based on surveys done between 1989–2013 every 3 years and we also interpolate the data within this period to obtain smoothed dynamics. Since the initial moment values are known at 1989, we fit the model parameters to the data. We perform two different fits and the results are presented in fig. 1:

- A constrained fit: We use the historical US average wealth and saved income ($\langle x_1(t)\rangle$ and $\langle x_2(t)\rangle$) available from ref. [13], to fit $\lambda_1 + \sigma_1^2/2$ and $\lambda_2 + \sigma_2^2/2$ for the period 1929–2010 using Ordinary Least Squares (OLS). We assume these parameters are valid during 1989–2013. Then we fit a single parameter —$\lambda_2 - \lambda_1$— to the measured CV values using OLS.
- An unconstrained fit: We use OLS to obtain σ_1, σ_2 and $\lambda_2 - \lambda_1$, without assuming any prior knowledge on $\lambda_1 + \sigma_1^2/2$ and $\lambda_2 + \sigma_2^2/2$.

The two different fits in fig. 1 differ considerably. The constrained fit only captures the general dynamics of the CV during 1989–2013, while the unconstrained fit better captures the transient during that period. This illustrates that the model is capable of capturing the complex dynamics of the CV. However, the parameters we obtain for the unconstrained fit cannot be used for prediction. We obtain $\lambda_2 - \lambda_1 = 0.35\,\text{year}^{-1}$. Such values are far from being representative of longer term wealth and income dynamics. In the constrained fit we obtain $\lambda_2 - \lambda_1 = 0.03\,\text{year}^{-1}$, which is a much more realistic value (see refs. [13,21]). We conclude that our model is mainly reliable for describing long term dynamics of wealth inequality, which is, indeed, our main goal. In the future, as more data on the US wealth CV becomes available, this could be further tested.

In general, assuming all parameters are positive, the coefficient of variation will eventually, after a sufficiently large time, grow exponentially, meaning that the distribution will indefinitely grow more unequal. Therefore, in order to obtain an insight on the possibility of wealth inequality reduction, we would look at a general initial state of the system, and test the conditions for the reduction of the $CV(t)$ at $t = 0$. The choice in $t = 0$ is only for the purpose of performing the calculation. Assuming the parameters and distributions are known for a given time, we can simply set this time to $t = 0$, considering the appropriate initial conditions.

Taking the time derivative of $CV(t)$, evaluated at $t = 0$ we obtain

$$\frac{\mathrm{d}CV(t)}{\mathrm{d}t}\Big|_{t=0} = \frac{-\langle x_2 (0)\rangle\langle x_1^2 (0)\rangle + \langle x_1 (0)\rangle\langle x_1 x_2 (0)\rangle}{\langle x_1 (0)\rangle^2\sqrt{\langle x_1^2 (0)\rangle - \langle x_1 (0)\rangle^2}} + \frac{\langle x_1^2 (0)\rangle\sigma_1^2}{2\langle x_1 (0)\rangle\sqrt{\langle x_1^2 (0)\rangle - \langle x_1 (0)\rangle^2}}. \tag{23}$$

Now, by solving $\mathrm{d}CV(t)/\mathrm{d}t|_{t=0} < 0$, it is possible to formulate a constraint on the initial conditions and the model parameters:

$$\langle x_1 x_2 (0)\rangle < \frac{\langle x_1^2 (0)\rangle\left(2\langle x_2 (0)\rangle - \sigma_1^2\langle x_1 (0)\rangle\right)}{2\langle x_1 (0)\rangle}. \tag{24}$$

If we define $\tilde{\beta} = \langle x_1(0)\rangle/\langle x_2(0)\rangle$, as the initial wealth–to–saved-income ratio, it is possible to rewrite the condition in eq. (24) as

$$\frac{\langle x_1 x_2 (0)\rangle}{\langle x_1^2 (0)\rangle} + \frac{\sigma_1^2}{2} < \frac{1}{\tilde{\beta}}. \tag{25}$$

The obtained constraint is consistent with empirical observations. Notably, low correlation between wealth and income is well known to be associated with a reduction of wealth inequality. When the savings rate is not very

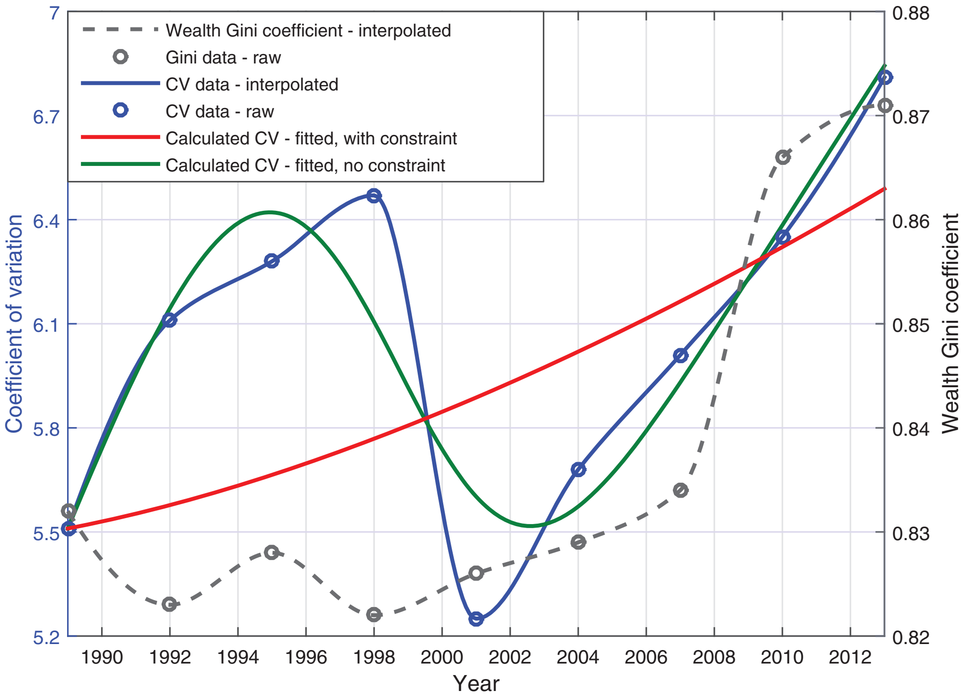

Fig. 1: (Color online) Model comparison with data. The coefficient of variation of the US wealth distribution (in blue circles, cubic interpolated in blue) 1989–2013 [6] and the theoretical coefficient of variation with best-fitted parameters, assuming a constraint on $\lambda_1 + \sigma_1^2/2$ and $\lambda_2 + \sigma_2^2/2$ from other data (red), and without the constraint (green). The measured wealth Gini coefficient is also presented in dark grey for reference (see ref. [21]).

low, the accumulation of wealth is mainly due to labor income, driving the wealth distribution to become closer to the income distribution, which is distributed more equally than wealth [3]. In addition, low σ_1^2 values reflect cases in which all individuals in the population experience a relatively similar wealth dynamics, and the relative wealth gap is only slowly formed. Interestingly, the other model parameters —λ_1, λ_2 and σ_2^2— may affect the rate in which the CV will change its trend from decreasing to increasing, but not whether it will initially increase or decrease. The reason is rather straightforward —if wealth and income increase at the same rate for all individuals, wealth inequality would not necessarily increase. Therefore, it is the variability of the paths of the different individuals that matters.

Figure 2 demonstrates the dramatic effect of σ_1 on the dynamics of $CV(t)$. It also illustrates, along with eq. (25), how the value of σ_1 required for the initial decrease of $CV(t)$ is dependent on $\tilde{\beta}$. These results echo the findings of Piketty [1] and others [3,10,22], showing that when the wealth–saved-income ratio is very high, and particularly when personal savings rates are low, wealth inequality is likely to increase, and vice versa. Since income is more equally distributed than wealth, a high savings rate strongly couples wealth to income, and therefore may lead to a low wealth-income ratio and to the reduction of wealth inequality, as explained in [4].

Inequality "relaxation time". – Based on the above derivation, a practical question can be asked: Given that according to eq. (24) wealth inequality indeed decreases at first and afterwards it increases ad infinitum – how much time will it take until the CV reaches its initial value again. This defines a characteristic time scale which we will refer to as the "relaxation time" of the CV and denote as t_{rel}.

When the constraint in eq. (24) does not hold, $t_{rel} = 0$. When it does, there is a practical importance to finding what is the scale of t_{rel}, whether it is days, years, decades or centuries. Using the full expression for $CV(t)$, it is possible to calculate t_{rel} and its dependence on the model parameters and initial values.

For that purpose we use the parameters obtained by the constrained fit presented in fig. 1. When the initial values that characterize 1929 are considered, we obtain $t_{rel} \approx 50$ years. The dependence of t_{rel} on various parameters is presented in fig. 3. It demonstrates first and foremost that t_{rel}, based on the estimated parameters, lies between 20 and 150 years. It shows, therefore, that there is a "natural" relaxation lasting up to several decades, assuming no external effects. These characteristic time scales are consistent with the data on wealth inequality. Following substantial changes in the economic and social structure and the aftermath of World War II, wealth inequality had been decreasing for several decades. Since the mid-1980s it is increasing in most Western countries, and is almost as high as it was during the 1920s (in the US;

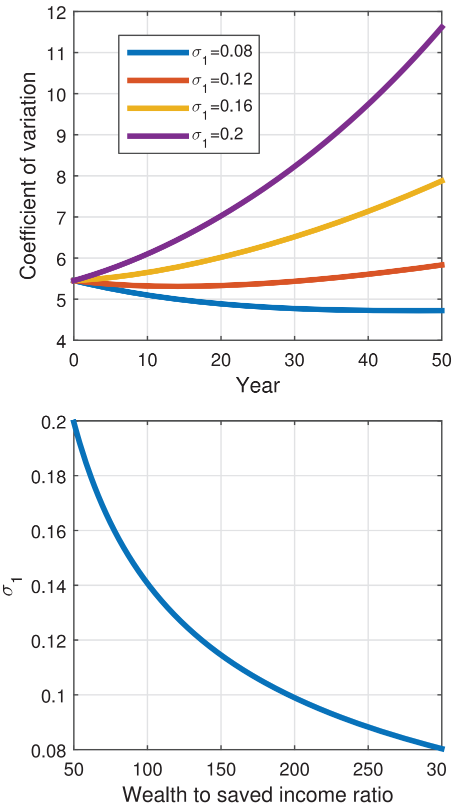

Fig. 2: (Color online) The effect of σ_1. $CV(t)$ for different σ_1 values (top). The dependence of σ_1 so that $\langle x_1 x_2(0)\rangle / \langle x_1^2(0)\rangle + \sigma_1^2/2 = 1/\tilde{\beta}$ holds, as a function of $\tilde{\beta}$, the wealth–to–saved-income ratio. The initial conditions representing 1989 were used (bottom).

in European countries the effect is less dramatic). This analysis also demonstrates that as $\tilde{\beta}$ increases, t_{rel} decreases. This supports the above observations on the effect of the wealth–saved-income ratio on inequality.

Discussion. – We devised a model for the dynamics of wealth and income of individuals, based on geometric Brownian motion. Using a Fokker-Planck equation, we were able to obtain a set of coupled differential equations for the dynamics of the joint income and wealth distribution moments, which we used for describing the dynamics of wealth inequality, quantified using the wealth distribution coefficient of variation. We take a "mean-field"-like approach by making several simplifications. Specifically, we assume that the noise terms of different individuals are independent, that the wealth and income growth rates are identical for each individual and ignore interactions between individuals. Despite these simplifications, we are still able to capture the essence of the dynamics, and hence capable of describing important aspects of the wealth distribution without providing a detailed description of the system. In that sense, the suggested model is a minimal model —further simplifying the model by neglecting the coupling between wealth and income cannot capture the wealth inequality dynamics.

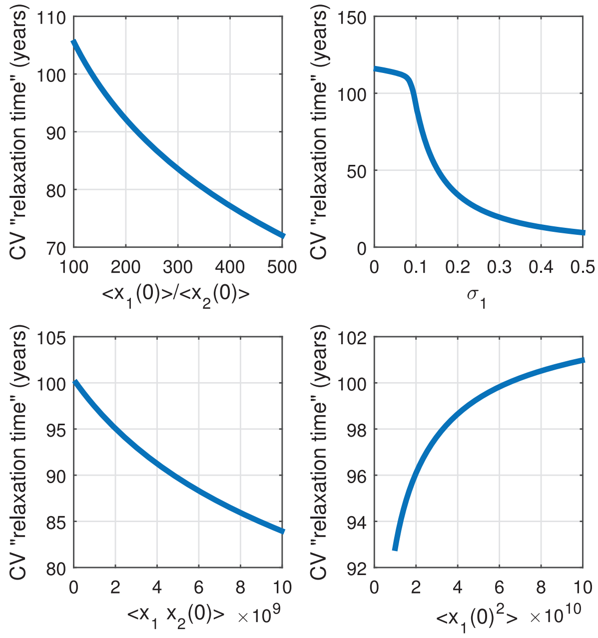

Fig. 3: (Color online) Inequality "relaxation time". The dependence of t_{rel} on different parameters and initial values: $\langle x_1(0)\rangle / \langle x_2(0)\rangle$ (top left), σ_1 (top right), $\langle x_1 x_2(0)\rangle$ (bottom left) and $\langle x_1^2(0)\rangle$ (bottom right). In all cases one parameter or initial value was changed and the rest were taken as characterizing the proper parameters and initial values for 1929 (see above). The values of t_{rel} are presented only for the parameter domain in which the constraint (24) holds.

The model differs from previous models of wealth dynamics by the explicit inclusion of stochastic terms and by analyzing the joint income-wealth dynamics, in contrast to the models presented in refs. [3,10,12]. This allows for obtaining closed-form expressions for the wealth distribution moments, hence new analytic results and insights.

Our main finding was that for any choice of parameters and initial conditions, wealth inequality eventually increases exponentially in time. However, under certain initial conditions, we found a closed-form criterion for the initial decrease of wealth inequality. This criterion can be fulfilled if the correlation between wealth and income is very low, the initial wealth inequality is very high, the average income is very high or the variability of wealth dynamics within the population is low (or a combination of these). These findings support the claim regarding the singularity of the 20th century in terms of wealth inequality dynamics.

Our model shows that wealth inequality will eventually increase in time regardless of the parameter values. As illustrated in the negligible saved-income case, in which we assume no coupling between wealth and income, the main parameter governing this increase is σ_1, the volatility of the wealth dynamics. The same effect was also demonstrated in other works [7,11,23]. In the case wealth inequality initially decreases, we can also quantify the "relaxation" time, t_{rel}, it takes the wealth coefficient of variation to return to its initial value. The relaxation time also depends on σ_1 and dramatically decreases as σ_1 increases.

We note that σ_1 is attributed partly to luck, but also incorporate all the differences between individual wealth dynamics affected by education, skill, background, efforts and so on. This result may seem at first as counter-intuitive, since the variability in the growth of wealth or income is intuitively parallel to mobility and it is widely accepted that the higher mobility is, the lower the inequality is [23]. We therefore cannot avoid the conclusion that as long as the economic system is free, the diverse ability of individuals to accumulate wealth will lead, in the long run, to the inevitable increase of wealth inequality. We should note, however, that the data on mobility concerns income, while we examine wealth.

In addition, we found that the increase of the wealth-income ratio will lead to a rapid increase in wealth inequality and vice versa. These results are consistent with most theories relating the wealth-income ratio to wealth inequality. This effect is originated in the limited correlation between wealth and income and since the income inequality is lower compared to wealth inequality [1,3,6].

More research should be carried out to establish the above: adding feedback terms to the model so that abrupt shocks may be explained within the model framework; considering non-linearity of the model parameters, *i.e.*, dependence on the wealth and income values; determining the effect of inheritance, population growth and economic mobility on wealth inequality; considering shocks involving changes in the model parameters through time and incorporating interactions between individuals.

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