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Revisiting $r > g$—The asymptotic dynamics of wealth inequality

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HIGHLIGHTS

- Wealth inequality is shown to become more and more inegalitarian if the capital value change rate exceeds the growth rate.
- The asymptotic behavior of the wealth distribution is only reached after a characteristic timescale of 100 years.
- The personal savings rate dominates the short timescale dynamics of the wealth inequality, but does not change the asymptotic wealth distribution shape.

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ABSTRACT

Studying the underlying mechanisms of wealth inequality dynamics is essential for its understanding and for policy aiming to regulate its level. We apply a heterogeneous non-interacting agent-based modeling approach, solved using iterated maps to model the dynamics of wealth inequality based on 3 parameters—the economic output growth rate g, the capital value change rate a and the personal savings rate s and show that for $a < g$ the wealth distribution reaches an asymptotic shape and becomes close to the income distribution. If $a > g$, the wealth distribution constantly becomes more and more inegalitarian. We also show that when $a < g$, wealth is asymptotically accumulated at the same rate as the economic output, which also implies that the wealth-disposable income ratio asymptotically converges to $s/(g - a)$.

1. Introduction

Wealth inequality has sharply increased during the past 30 years in most western countries leading to research of the sources and factors contributing to the dynamics of the wealth distribution. A particularly important explanation for the current trend, which received interest both within and outside academia, is Piketty's so-called $r > g$ argument [1]. Historically, the rate of return on private capital r exceeded the growth rate of the economy g and Piketty argues that while $r > g$, the wealth of the capitalist class will eventually grow faster than the incomes of workers, leading to an "endless inegalitarian spiral" [1,2].

This argument garnered both support and criticism. Madsen et al. [3] presented findings which verify some of the observations made by Piketty to justify his argument. Solow [4] supported most of Piketty's arguments, but cast doubt on the claim that as the wealth-income ratio is likely to increase, the return on capital will increase as well, which is a fundamental assumption in Piketty's explanation. A similar critique was repeated by Mankiw [2], Rognlie [5] and Kanbur and Stiglitz [6], suggesting mainly that "a rising share of capital can be consistent with the other stylized fact of rising capital-output ratio

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only if the elasticity of substitution between capital and labor is greater than unity, which is not consistent with the broad empirical findings" [6].

Rogoff [7] favorably reviewed Piketty's arguments, partially supporting Piketty's $r > g$ argument, while stating that "Piketty and Saez do not really offer a model; nor does this new book. And the lack of a model, combined with a focus on the world's upper-middle-class countries, matters a lot when it comes to policy prescriptions".

The wealth-income ratio plays an important role in the theory of wealth inequality, as "high steady-state wealth-income ratios can go together with large instability, asset price bubbles, and high degrees of inequality" [8]. According to the Harrod–Domar–Solow wealth-income ratio formula [8–12], the steady-state wealth-income ratio asymptotically follows $\beta = s/g$, where s denotes the personal savings rate. In addition, according to various classes of macroeconomic models, this ratio is considered to be a decreasing function of the growth rate [8,12]. This further implies that economic variables that affect inequality, such as the personal savings rate s and the economic growth rate g, will also determine the wealth-income ratio.

The purpose of this paper is to use a novel framework for quantitatively evaluating the asymptotic behaviors of wealth inequality and the wealth-income ratio and discuss the role growth and return on capital play on these behaviors. The presented framework significantly differs from the traditional macroeconomic methodologies used for analyzing such effects, as we do not aim to determine the plausibility of the different assumptions made by Piketty and other scholars regarding the capital–income ratio. In particular, we do not incorporate explicitly the elasticity of substitution between capital and labor, or assume the existence of an equilibrium. For our analysis we consider a system in which the capital value change rate a, the personal savings rate s and the economic output growth rate g are fixed and study its dynamics using a heterogeneous non-interacting agent-based model. We seek to reveal whether such a system reaches an equilibrium, in which a stable wealth distribution is obtained, and how the different parameters, *i.e.* the return on capital, the personal savings rate and the growth rate, affect this distribution and the steady state wealth-income ratio. The understanding of the idealized system is then used for making general conclusions on realistic economies.

2. The modeling framework

Our analysis is based on the model presented in Ref. [13]. It describes the wealth accumulation processes of individuals within a population based on 3 sources:

1. Labor income—accounts for the income originated from wages and earnings. Only a fraction of the labor income contributes to wealth, due to taxation and spending.
2. Capital income—accounts for the income originated from wealth, including profits, royalties, rents and dividends. Only a fraction of it contributes to the accumulated wealth, due to taxation and spending. Capital and labor incomes together constitute the total national income.
3. Capital value change—accounts for the value change of capital such as owned land, estates, shares, options and other assets owned by individuals. Some of the capital–income, a part of the national income, is invested back, contributing to an increasing value of assets, accounted therefore as capital value change and not as capital–income.

The model was introduced in Ref. [13], and is described in detail in Appendix A. We consider a population of N agents and assume that for each agent i the private wealth $W_i(n)$ and disposable income $D_i(n)$ are governed by the following equations:

$$\begin{aligned} D_i(n+1) &= D_i(n)\left[1+g(n)\right] \\ W_i(n+1) &= W_i(n)\left[1+a_i(n)\right] + s_i(n) D_i(n), \end{aligned} \tag{1}$$

where $a_i(n)$ is the effective capital value change rate of i and $s_i(n)$ is the savings rate of i. We also denote $W(n) = \sum_{i=1}^{N} W_i(n)$ and $D(n) = \sum_{i=1}^{N} D_i(n)$ as the total private wealth and total disposable income, respectively.

We note that a is significantly different from Piketty's definition of r as the effective rate of return on capital, since we separate capital–income, which is incorporated as a part of the disposable income and the capital value change. $D(n)$ progresses in time according to the growth rate g. There is a difference between the GDP growth rate and the growth rate of the disposable income, however it is very small in practice (see Appendix B for more details).

The historical private wealth and disposable income, along with the values of g, a and s are presented in Fig. 1. This figure shows that historically, wealth and income were growing at a very similar rate. In addition, it is found that $g > a$ during most of the period in discussion with average values of $\bar{g} = 3.2\%\ \text{year}^{-1}$ and $\bar{a} = 1.5\%\ \text{year}^{-1}$. a and g are also positively correlated, with a Pearson correlation coefficient of 0.42.

Eq. (1) can be reduced to the individual level, enabling us to propagate the wealth of each individual in time, based on the different parameters (a, g and s). This way the wealth distribution can be obtained and the wealth inequality – quantified by the wealth share of the top 10% – calculated. The historical top 10% wealth share in the US is presented in Fig. 2 along with the model results, based on the historical behavior of the parameters. These results were presented originally in Ref. [13].

When applied to the individual level, the savings rate s and the capital value change a depend on income and wealth, respectively. In other words, we assume that s increases with income and that a increases with wealth (see Appendix A). This dependence is not only realistic, as individuals with higher incomes save a larger fraction of their income and wealthier

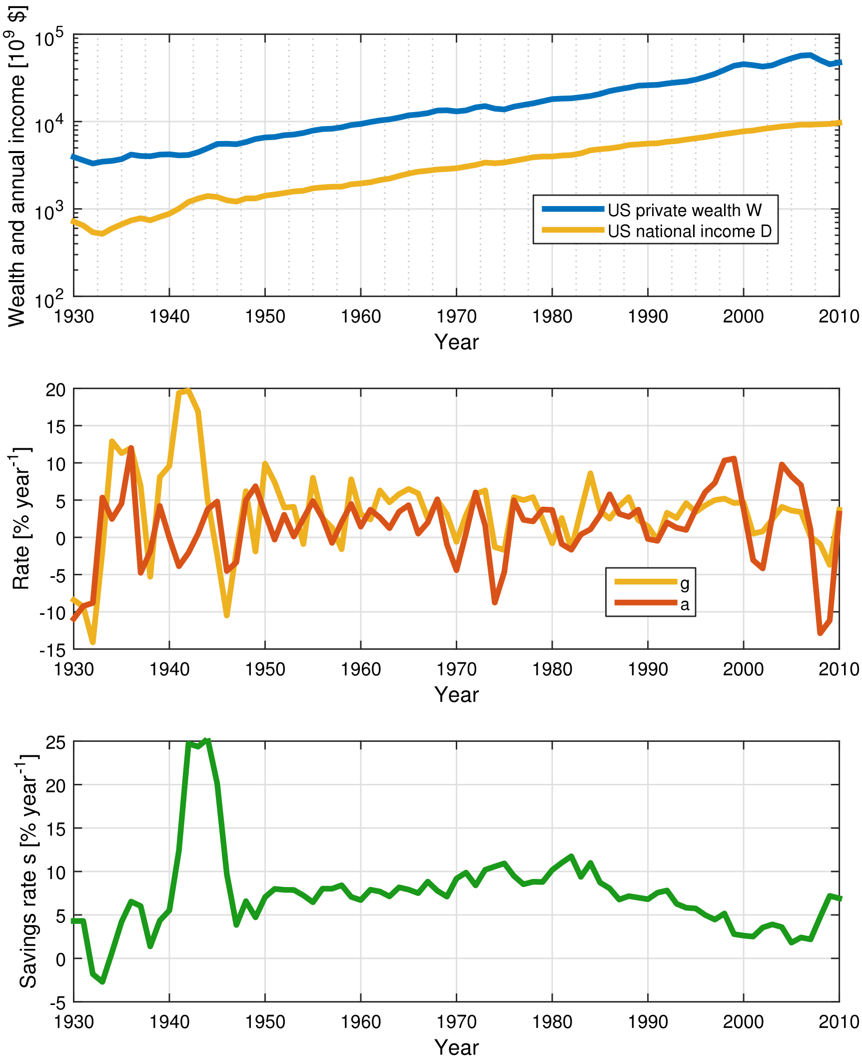

Fig. 1. Historical wealth and income in the US. Top: The historical private wealth (blue) and national income (yellow) in the US [8]. The values are inflation adjusted. Middle: The historical real national income growth rate (yellow) and calculated value of a (orange) [8,13]. Bottom: The historical personal savings rate in the US [8]. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

individuals get higher returns on their investments [1,13,14], but also essential for proper modeling of wealth accumulation processes. For uniform a and s, the wealth distribution would remain approximately constant in time. For more details on the dependence of s and a on income and wealth, the reader is referred to Appendix A and to Ref. [13].

We also note that a and g can be negative, in general, but only for relatively short periods of time. When averaged over decades or a few centuries, their values are positive in all western countries. Several years of negative growth, or negative value of a, will have no significant effect on the model outcome in the long run, which is essentially dictated by the average value of the parameters (refer to Appendix C for more details).

3. Results

Using the presented model, and based on the good agreement of the model results with the historical behavior of the wealth inequality, it is now possible to test the asymptotic behavior of the wealth distribution with respect to a, g and s. In order to perform this test, we use numeric simulations in which individuals are given an initial income and wealth, according to the realistic distributions of income and wealth in 1930 in the US. The income and wealth of each individual are then iteratively propagated in time according to Eq. (1), while every time step accounts for one year. For each year we then calculate the top 10% wealth share from the obtained wealth distribution. For more details on the implementation of the model, refer to Appendix A and to Ref. [13].

We performed such calculations for various cases in which different values of a, g and s were considered. In all calculations the set of parameters was kept constant in time. We obtained convergence of the top 10% wealth share and

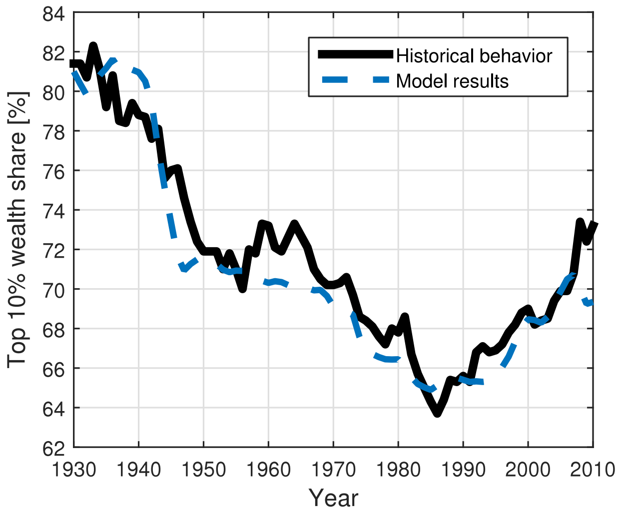

Fig. 2. Historical wealth inequality in the US. The historical wealth share of the top 10% in the US (black) and the model results (dashed blue) based on the historical values of the parameters [13].

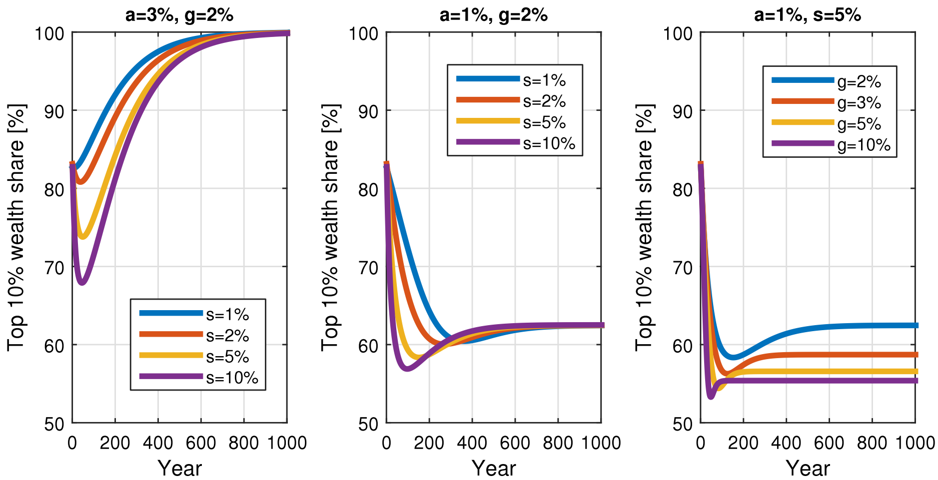

Fig. 3. The top 10% wealth share asymptotic behavior. The model results for the top 10% wealth share were calculated for several different cases.

terminated the calculation when the relative difference between subsequent wealth shares was smaller than 10^{-7} (the results were insensitive to the convergence criterion).

The results for different values of a, g and s are presented in Fig. 3. They confirm that the shape of the distribution reaches a steady state in the sense that the top 10% wealth share converges to an asymptotic value.

Based on Fig. 3 it is possible to make several observations:

- The asymptotic wealth distribution shape is determined by the relation of a and g and is independent on the value of s (the expected wealth will be dependent on s). When $a > g$, the wealth distribution asymptotically becomes perfectly unequal, reaching a top 10% wealth share of 100%. If $a < g$, the wealth distribution gets closer to the income distribution and the wealth share of the top decile reaches an asymptotic value, lower than 100%. It becomes lower, and closer in shape to the initial income distribution, as g gets larger. This observation is supported by looking at the wealth and income distributions in detail, as presented in Fig. 4.
- The time required for changes in parameters to be fully absorbed in the wealth distribution – and therefore required for it to reach a steady-state – is usually more than 100 years. It implies that in reality the wealth distribution is far from equilibrium, since changes in the parameters that determine the wealth distribution are much more rapid, in practice.
- The savings rate does not affect the asymptotic wealth distribution shape, but it dominates the dynamics. When $g > a$, convergence is reached faster as the savings rate increases and vice versa when $a > g$. Hence, for practical estimations and predictions of wealth inequality in short time horizons, the savings play a major role, which might have a larger effect on wealth inequality than g and a. This echoes findings on the importance of the personal savings rate to wealth inequality [13,15,16].

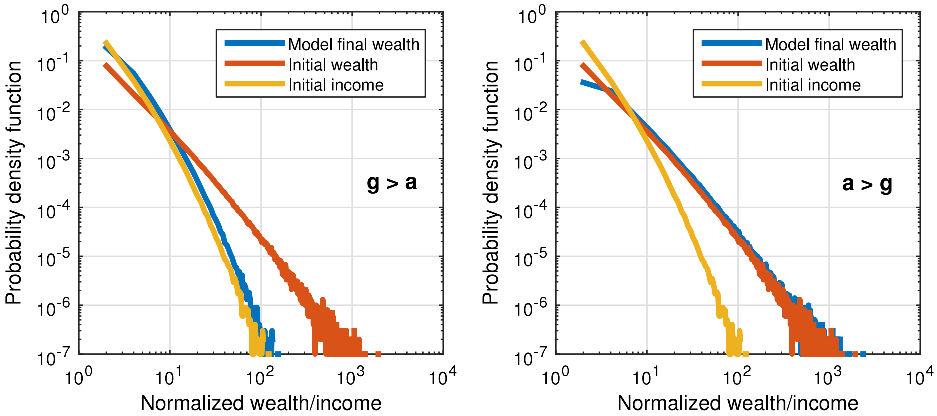

Fig. 4. The income and wealth distributions. The population initial distribution of income (yellow) and wealth (orange) along with the asymptotic wealth distribution produced by the model (blue). The distributions were normalized so that their mean value is 1 for presentation purposes. Left: $g > a$, $s = 5\%$, $a = 1\%$ and $g = 10\%$. Right: $a > g$, $s = 5\%$, $a = 10\%$ and $g = 1\%$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

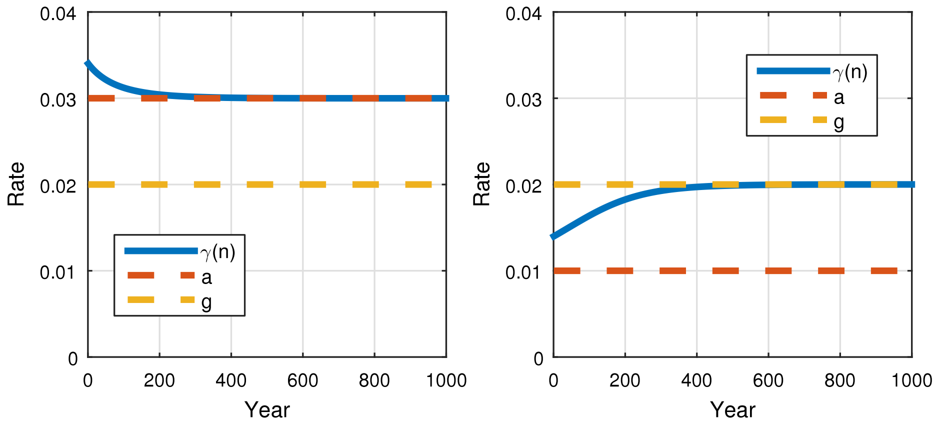

Fig. 5. The dynamics of the wealth accumulation rate. The wealth accumulation rate γ (see Eq. (2)) behavior in time (blue). a (orange) and g (yellow) were kept constant for the whole time, and the savings rate considered was 2% year^{-1}. The savings rate does not affect the asymptotic value of γ, but only its initial value and the speed in which it converges into the g or a. Left: $a > g$. Right: $g > a$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The personal savings rate controls the short-term dynamics of the system and as the savings rate gets larger, there would be a deeper initial dip in the top 10% wealth share. This dip can be attributed to the process in which the wealth distribution gradually becomes closer to the income distribution. This process is accelerated as the savings rate is larger. Appendix D presents the dependence of the top 10% time derivative as a function of the personal savings rate. If $a > g$, however, this effect is rapidly attenuated, since wealth increases substantially faster than income. This can be explained by the wealth accumulation rate, defined as follows, based on Eq. (1):

$$\gamma(n) \equiv \frac{W(n+1) - W(n)}{W(n)} = a + s\frac{D(n)}{W(n)}. \tag{2}$$

When considered asymptotically Eq. (2) becomes (refer to Appendix E for more details):

$$\gamma(n) \xrightarrow{n\to\infty} \begin{cases} a, & g < a \\ g, & a < g. \end{cases} \tag{3}$$

If $a > g$, the income becomes very small when compared to the wealth, the ratio $D(n)/W(n)$ converges to 0 and γ approaches a. However, if $g > a$, we obtain that asymptotically γ approaches g. This is demonstrated in Fig. 5. This result is consistent with the behavior depicted in Fig. 1, showing that the growth rate is similar to the rate of wealth accumulation, while during most of the period in discussion, $g > a$. In addition, this demonstrates that the choice in a is relevant for practical purposes, and helps to further validate our modeling framework.

We note that while the convergence of the top 10% wealth share, as presented in Fig. 3, is relatively slow, the convergence of the wealth accumulation rate, depicted in Fig. 5, is usually faster and is strongly dependent on the initial value of

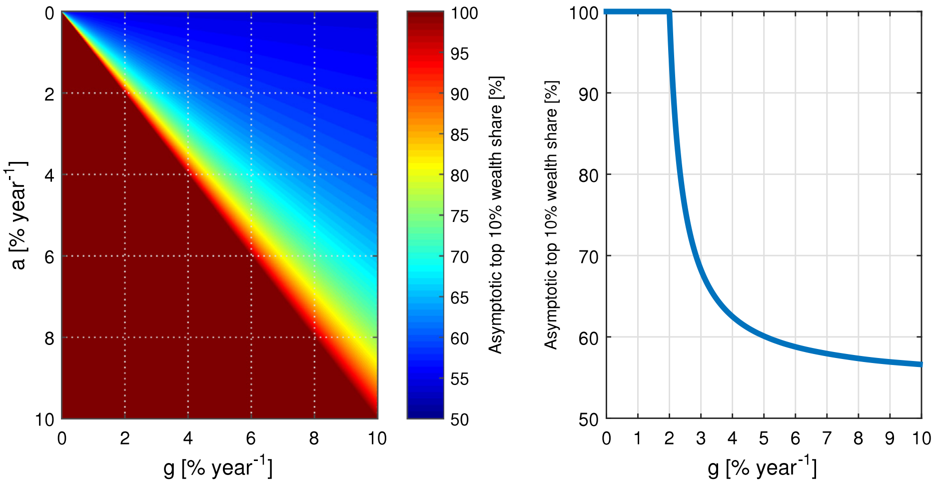

Fig. 6. The dependence of the asymptotic top 10% wealth share on a and g. The asymptotic value the top 10% wealth share converges to depends on a and g. Left: Heat map of the asymptotic value the top 10% wealth share with a and g running from 0% to 10%. Right: The dependence of the asymptotic top 10% wealth share on g given $a = 2\%$.

wealth-income ratio. For example, given an initial wealth-income ratio that is 50% larger than the asymptotic value $s/(g-a)$ (see 3.1), the convergence would take about 20 years, compared to the slow convergence depicted in Fig. 5, in which the initial wealth-income ratio was 5 times larger than the asymptotic value. For that reason, the average wealth accumulation rate in the US between 1930 and 2010 is very similar to the average growth rate during the same period (see Eq. (4)).

While Fig. 3 shows that the asymptotic top 10% wealth share depends on a and g, Fig. 6 depicts the same dependence, showing that as long as $a > g$, the asymptotic top 10% wealth share is 100%. When $g > a$ the asymptotic top 10% wealth share is approximately inversely proportional to g. The asymptotic wealth share has a lower bound—the top 10% wealth share of the income distribution, which was taken as 42% [17]. We note that the income distribution is taken as constant in time, while in practice it changes considerably [17]. However, it affects the behaviors observed only quantitatively and not qualitatively. If follows from the results that if $g > a$ and the initial income distribution is not much different from the initial wealth distribution, then the top 10% wealth share will not change significantly in time. A direct practical implication is that the growing income inequality observed in several western economies, though not necessarily fuels the surge in wealth inequality, indeed limits the ability to control or reverse this surge.

3.1. The asymptotic wealth-income ratio

A direct corollary of the previous results is the existence of a finite asymptotic wealth-disposable income ratio when $g > a$. Since asymptotically wealth and income grow at the same rate in such a case, this ratio can be simply calculated to obtain (see Appendix E for proof):

$$\beta' = \frac{W}{D} = \frac{s}{g-a}. \tag{4}$$

This result differs from the result of the Harrod–Domar model, in which the steady state wealth-income ratio follows $\beta = s/g$ [8–11]. Considering, for example, the case in which $s = 6\%\ \text{year}^{-1}$, $g = 3\%\ \text{year}^{-1}$ and $a = 2\%\ \text{year}^{-1}$, the steady state wealth-disposable income ratio would be 600%, while according to the Harrod–Domar–Solow formula it would be 200%. The historical values of β in the US range between 350%–400% during 1950–1970, to 400%–500% during 2000–2010 [8]. When the disposable income is considered and not the total national income, these values increase to 450%–650%. For the period 1946–2010, the average values of g, a and s in the US, were $3.09\%\ \text{year}^{-1}$, $1.96\%\ \text{year}^{-1}$ and $7.11\%\ \text{year}^{-1}$, respectively, corresponding to an asymptotic wealth-disposable income ratio of 630%.

We note again that g considered in our framework is the disposable income growth rate and not the national income or GDP growth rate, which slightly differs from one another (see Appendix B). Considering $s/(g-a)$ with the real national income growth rate would correspond to an asymptotic wealth-income ratio of 700% for the period 1946–2010. Considering s/g for the same period and the real national income growth rate corresponds to an asymptotic wealth-income ratio of 240%, which is substantially lower from the historical characteristic values of β.

The resulting realistic values and the empirical evidence $g \approx \gamma$ (see Fig. 1) demonstrate the validity of Eq. (4). However, its validity should also be tested using standard equilibrium macroeconomic models, which exceeds the scope of this work.

4. Discussion

We used a heterogeneous non-interacting agent-based model for studying the dynamics of wealth inequality and analyzing its asymptotic behavior depending on 3 main parameters—the economic output growth rate g, the capital value change rate a and the personal savings rate s.

The value of a is derived from the change in wealth, accounting for changes in owned capital value not considered as any sort of income. For the US economy, a was historically comparable to the growth rate but almost consistently lower. We found that given $a > g$, the wealth distribution will become constantly more and more inegalitarian. However, if $a < g$, the wealth distribution gets closer to the income distribution, hence becomes more equal. In this situation, the wealth share of the top decile reaches an asymptotic value, lower than 100%. In any case, a steady state distribution shape was obtained.

The wealth accumulation rate γ, defined as the annual rate in which the total wealth changes during a given year, was found to asymptotically follow g, if $a < g$, and a, otherwise. This is consistent with the realistic case in which $g > a$ during most of the 20th century, and the average value of γ is 3.1% year^{-1}, while $\bar{g} = 3.2\%$ year^{-1} and $\bar{a} = 1.5\%$ year^{-1}. This result implies that a finite asymptotic wealth-disposable income ratio exists when $a < g$ and equals $\beta' = s/(g-a)$, which is related to the traditional Harrod–Domar–Solow formula for the steady state wealth-income ratio $\beta = s/g$ [8–11].

In addition, it was found that the time required for changes in parameters to be fully absorbed in the wealth distribution, reaching a steady-state, is usually more than 100 years. It implies that in practice, the wealth distribution is far from equilibrium, since changes in the parameters that determine the wealth distribution are much more rapid (see also Refs. [18,19]). The speed with which the steady state is obtained is largely determined by the personal savings rate. The savings rate does not affect the asymptotic wealth distribution shape, while it dominates the dynamics of the distribution. Hence, savings play a major role in determining future wealth inequality in realistic time scales of up to several decades.

Our findings help to further validate the presented modeling framework, supporting some of the arguments made by Piketty and others regarding the long term behavior of the wealth inequality. We also use the model to demonstrate once again the importance of transients rather than asymptotic values or equilibria in the context of wealth inequality, due to the long characteristic time scales of its dynamics. This highlights the significance of personal savings in controlling these dynamics, most notably for practical purposes and short time horizons. The direct, most relevant implication of this result is that a significant increase in personal savings is likely to lead to lower wealth inequality in a substantial manner compared to an increase in the economy growth rate, see also Ref. [15].

Acknowledgments

We thank Thomas Piketty for his feedback on the manuscript. We also thank an anonymous referee for thoughtful and constructive comments and suggestions.

Appendix A. The model

The model used for the analysis is a heterogeneous non-interacting agent-based model describing the dynamics of wealth inequality, described and analyzed in Ref. [13]. It implements wealth accumulation of individuals within a population based on 3 sources – labor income, capital–income and capital value change – as described in Section 2. We now consider a population of N agents, in which each agent i is characterized at discrete time steps n, by the wealth it owns $W_i(n)$ and by its disposable income $D_i(n)$. The time interval between consecutive time steps is taken as one year, due to the availability of data. The initial values were taken from Piketty and Zucman [8] and from Wolff [14], and reflect the actual personal wealth and disposable income distributions in the US in 1930. The wealth and income initial distributions were created so that a correlation of 0.55 between them is maintained, similar to the correlation found in practice [20].

Let us denote $W(n) = \sum_{i=1}^{N} W_i(n)$, as the total personal wealth and $D(n) = \sum_{i=1}^{N} D_i(n)$ as the total disposable income at time step n. Following these notations, the total personal wealth at time $n+1$ is equal to the total personal wealth at time n added to the capital value change, denoted as $R(n)$, and to the income contribution to wealth. This contribution is calculated by taking into account the average personal savings rate from disposable income $s(n)$. This equality can be put into the following equation:

$$W(n+1) = W(n) + R(n) + s(n) D(n). \tag{A.1}$$

In practice, the different parameters and variables are time dependent. Within the scope of this paper, we consider $s(n)$, $a(n) = R(n)/W(n)$ and the growth rate $g(n) = (D(n+1) - D(n))/D(n)$ to be constant. We denote the wealth value change rate by a to distinguish it from the rate of return on capital as presented by Piketty [1] and others [16,21]

In reality, the historical values of all variables in Eq. (A.1) can be extracted from various sources [8,22,23], apart from $R(n)$, which is calculated simply by solving the equation using the extracted data. The model analysis for the historical case is presented in Ref. [13].

Eq. (A.1) can be now used for describing the accumulation of wealth given a single individual. Solving this equation for all individuals will enable us to calculate the distribution of wealth at each time step, hence the wealth inequality.

The dependence of savings on income is due to the tendency of individuals with higher income to save a larger fraction of it [24–28]. We assume this dependence is constant in time, though in fact it changed through history. However, the model

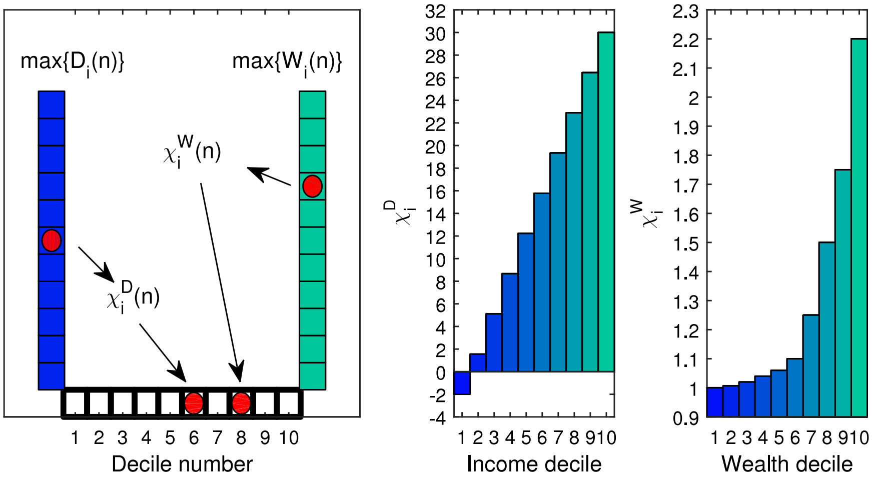

Fig. A.7. The division of wealth and income to deciles. Left: An illustration of the division to deciles—every time step n, each individual, with disposable income D_i and wealth W_i is attached to a certain decile in income and wealth according to the distribution of the entire population. Middle: The dependence of the disposable income fraction, χ_i^D, on the income decile. Right: The dependence of χ_i^W on the wealth decile. The presented values of χ_i^D and χ_i^W are not normalized. In practice the values are normalized so that $\sum_{i=1}^{N} s(n)\,\chi_i^D(n)\,D_i(n) = s(n)\,D(n)$ and $\sum_{i=1}^{N} a(n)\,\chi_i^W(n)\,W_i(n) = a(n)\,W(n)$.
Source: This figure is taken from Ref. [13].

results are not very sensitive to such changes [13]. In order to consider this dependence, we divide the population to income deciles, and multiply the income of each individual by a corresponding factor $\chi_i^D(n)$, which satisfies $\sum_{i=1}^{N} \chi_i^D(n)\,D_i(n) = D(n)$ and hence also

$$\sum_{i=1}^{N} s(n)\,\chi_i^D(n)\,D_i(n) = s(n)\,D(n)\,. \tag{A.2}$$

Additionally, we consider the dependence of capital value change on wealth, divide the population to wealth deciles, and multiply the wealth of each individual by a corresponding factor $\chi_i^W(n)$, which satisfies $\sum_{i=1}^{N} \chi_i^W(n)\,W_i(n) = W(n)$ and hence

$$\sum_{i=1}^{N} a(n)\,\chi_i^W(n)\,W_i(n) = a(n)\,W(n) = R(n)\,. \tag{A.3}$$

The dependence of χ_i^W on wealth is mainly due to different asset classes owned by wealthier individuals compared to poorer individuals [21,29], in addition to better terms of investment and investment possibilities accessible to rich individuals, but inaccessible to the poor [1,30,31]. We assume this dependence is constant in time and is constructed by the data provided by Wolff [29], and set the highest value of χ_i^W to be 2.2 times more than the lowest value. The model results are not very sensitive to this choice [13], however, if no dependence is taken into account (meaning that $\chi_i^W = \chi^W,\ \forall i$), no increase in wealth inequality can be possible based on capital accumulation. In such case, the share of wealth owned by the top deciles will not increase and will only decrease or stay unchanged. A typical dependence of χ_i^D and χ_i^W on income and wealth is depicted in Fig. A.7.

Based on the above, and on Eq. (A.1) we can formulate an equation for the wealth accumulation of each individual i:

$$W_i(n+1) = W_i(n) + \chi_i^W(n)\,a(n)\,W_i(n) + \chi_i^D(n)\,s(n)\,D_i(n)\,. \tag{A.4}$$

Eq. (A.4) is clearly consistent with Eq. (A.1), meaning that summation over $W_i(n+1)$ for all individuals results in Eq. (A.1). By solving the iterated map for all individuals, we obtain the wealth distribution at each time step. We note that in order to properly propagate the labor income in time, we consider the relation $D_i(n+1) = D_i(n)\,(1 + g(n))$. Following the obtained distribution, we calculate the share of wealth owned by the top wealth decile. This measure will be used as the primary measure for wealth inequality within the scope of this work.

Appendix B. The real GDP and disposable income growth rates

Fig. B.8 depicts the historical real GDP and disposable income growth rates in the US between 1930 and 2010. It demonstrates that their behavior in time is almost identical and the differences between them are very small.

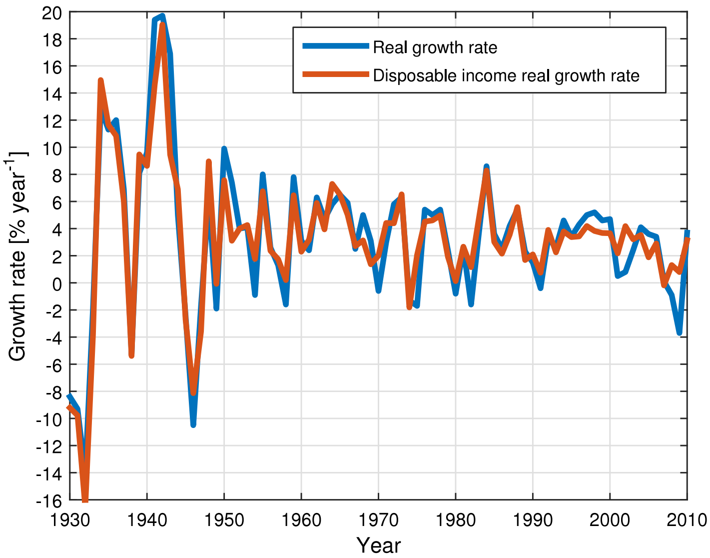

Fig. B.8. The real GDP and disposable income growth rates. The real GDP growth rate (blue) and the disposable income growth (orange) in the US, 1930–2010. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
Source: Data are taken from Ref. [17].

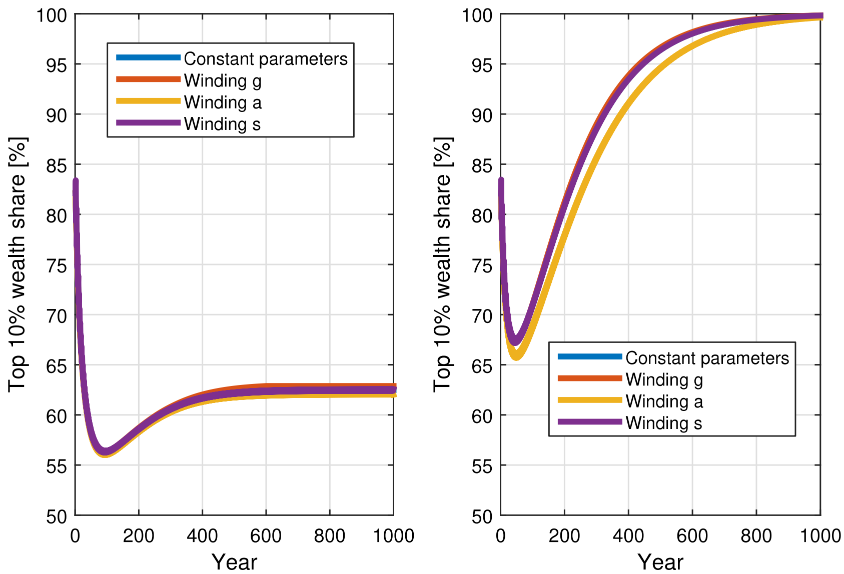

Fig. C.9. The effect of negative parameters on the model results. The top 10% wealth share asymptotic behavior for two cases—Left: the average parameters are $g = 2\%\ \text{year}^{-1}$, $a = 1\%\ \text{year}^{-1}$ and $s = 10\%\ \text{year}^{-1}$. Each curve presents the results with a different parameter changing between $3x$ to $-x$, where x is its average value. For example, for the red curve, if in a given year g is 6%, in the consecutive year it is −2%, and then again 6% and so on and so forth, so that the average value remains 2%. Right: similar to the left figure, with the average parameters $g = 2\%\ \text{year}^{-1}$, $a = 3\%\ \text{year}^{-1}$ and $s = 10\%\ \text{year}^{-1}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Appendix C. The effect of negative and average parameter values

Since in practice the values of a, g and s can be negative for periods of several years, we wish to demonstrate that the model results are robust even when extreme changes between consecutive years occur in these parameters, as long as their average value remains the same. These results are illustrated in Fig. C.9.

In addition, we also wish to demonstrate that the long run dynamics of the top 10% wealth share are governed by the average parameter values. For that purpose we considered the historical values of a, g and s in the US between 1930 and 2010 [1]. For each parameter we the calculated average and the standard deviation in this period. Assuming the parameters were normally distributed, we created multiple independent realizations of the model. In each realization a, g and s were randomly drawn every time step (year) given these distributions. This created multiple trajectories of the top 10% wealth

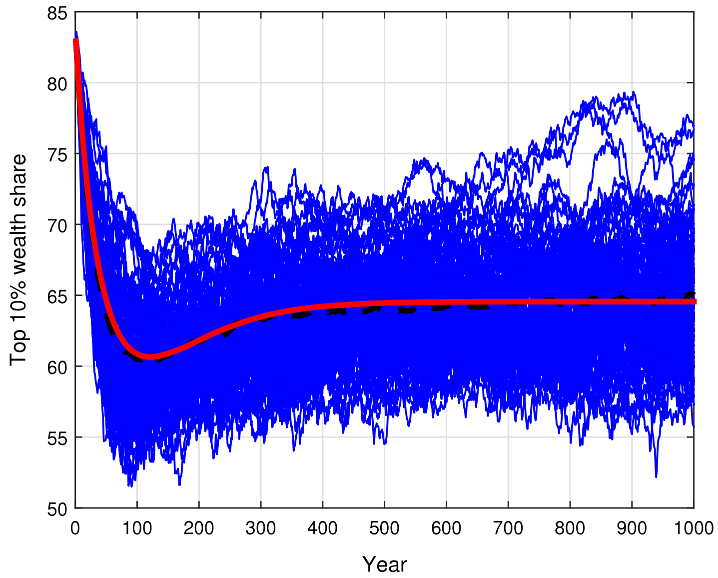

Fig. C.10. The effect of average versus random parameter values on the model results. The top 10% wealth share for 100 independent realizations of the model, considering randomly drawn a, g and s following the distributions $\mathcal{N}(0.015, 0.0023)$, $\mathcal{N}(0.032, 0.0026)$ and $\mathcal{N}(0.056, 0.0013)$, respectively (thin blue curves). The thick dashed black curve is the average of the 100 realizations. The thick red curve is the model result for the top 10% wealth share assuming a, g and s were fixed for the entire period to their average values 1.5% year^{-1}, 3.2% year^{-1} and 5.6% year^{-1}, respectively.

share which are randomly distributed around the model result, given the fixed averaged parameter values. This is presented in Fig. C.10.

Appendix D. The effect of savings

As demonstrated above, the savings rate does not affect the asymptotic wealth distribution. However, in short time horizons the savings play a major role in the dynamics of the wealth distribution, which might have a larger effect on wealth inequality than g and a. As the savings rate gets larger, an initial dip in the top 10% wealth share gets sharper (see Fig. 3). The dip is originated, as explained, in the lower inequality characterizing the income inequality. When the savings rate increases, the part of the income in the progression of wealth becomes larger, contributing to this decrease in wealth inequality. Fig. D.11 presents the dependence of the top 10% time derivative as a function of the personal savings rate, for the initial time step of the simulation, demonstrating that the initial dip in wealth inequality becomes linearly larger with the savings rate.

Appendix E. The wealth accumulation rate

The continuous form of the iterative map Eq. (1) can be written as

$$\dot{W} = aW + D_0 s\, e^{gt}, \tag{E.1}$$

where D_0 is the initial total disposable income, s is the average personal savings rate and a and g are the capital value change rate and the growth rate, respectively. We note, however, that the definitions of a and g are slightly different than those in Eq. (1), since in the iterative map, we use discrete time steps and annual rates and not continuous time.

The solution of Eq. (E.1) is

$$W = \frac{D_0 s \left(e^{gt} - e^{at}\right)}{g - a} + W_0 e^{at}, \tag{E.2}$$

assuming s, a and g are constant and W_0 is the initial total private wealth.

From Eq. (E.2) it follows that asymptotically $W/W_0 \sim e^{at}$ if $a > g$ and $W/W_0 \sim e^{gt}$ if $a < g$. This is trivially applicable for Eq. (1), implying the γ, the wealth accumulation rate, asymptotically follows g if $g > a$ or a, if $a > g$.

This result is illustrated in Fig. 5 and is also consistent with the realistic case in which $g > a$ during most of the 20th century. This historical average value of γ was 3.1% year^{-1}, while $\bar{g} = 3.2$% year^{-1} and $\bar{a} = 1.5$% year^{-1}.

The asymptotic wealth-income ratio can also be obtained from Eq. (E.2), if $a < g$. In this case we obtain that $W \approx (D_0 s)\,/\,(g - a) \cdot e^{gt}$ asymptotically, while $D = D_0 e^{gt}$. Therefore we obtain:

$$\frac{W}{D} \approx \frac{s}{g - a}. \tag{E.3}$$

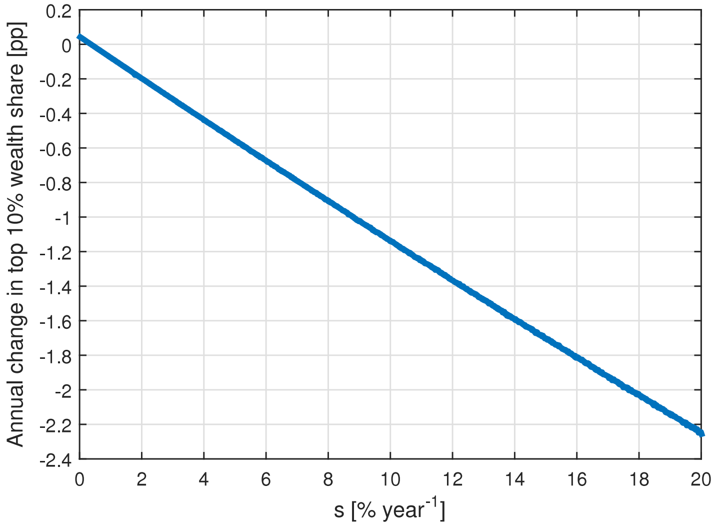

Fig. D.11. The dependence of the change in the top 10% wealth share on the savings rate. The change between two consecutive years in the top 10% wealth share is linearly dependent on the wealth share. Every average 1% of savings decreases the wealth share by approximately 0.1% percentage points. This result is valid only for the initial value considered (corresponding to the wealth and income distributions in the US in 1930).

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